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Irreducible Representation of the Quantum Group $E_q(2)$

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A basis for an irreducible representation of the quantum algebra $E_q(2)$ is given, consisting of eigenfunctions of the q-differential representation of the Casimir operator of the quantum algebra $E_q(2)$.

1. INTRODUCTION

Lie theory gives a natural setting for an algebraic interpretation of the special functions [1, 2]. Using the exponential mapping from the algebra to the corresponding group, one computes the matrix elements of group operators in specific irreducible representations and finds that these are typically expressible in terms of special functions. As an example consider the group $SU(2)$ with diagonal subgroup isomorphic to $U(1)$. The matrix elements of the irreducible representations of $SU(2)$ with respect to $U(1)$ basis can be expressed in terms of Jacobi polynomials and the spherical functions are the associated Legendre polynomials.

For quantum groups the situation is different. Only few quantum subgroups are available [3–5]. A similar connection has been established [6–8] between quantum algebra and the so-called basic or q-special functions. In this case one considers matrix elements of operators built with q-exponentials of the algebra generators; these elements turn out to be expressible in terms of q-hypergeometric series. Also, q-special functions appear as bases of irreducible representations of quantum algebras. Two anlages of the exponential play an important role in our approach. They are defined by

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$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \qquad |z| < 1 \tag{1}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{[n]_q!} z^n \tag{2}$$

where $[n]_q = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$ and $[n]_q! = [n]_q\,[n-1]_q \ldots [1]_q$.

The two exponential functions have the following properties:

1. $\lim_{q\to 1^-} e_q(z(1-q)) = \lim_{q\to 1^-} E_q(z(1-q)) = \exp(z)$.
2. $E_q(x+y) = E_q(x)E_q(y) = E_{q^{-1}}(x)E_{q^{-1}}(y)$, such that $xy = qyx$.
3. $e_q(z_1)e_q(z_2) = e_q(q^{-N_2/2}z_1 + q^{N_1/2}z_2)\cdot$, where the operators $N_i = z_i\partial_i$, $i = 1, 2$, act on the constant 1, indicated by "·".
4. $e_q(z)E_q(-z) = 1$
5. $E_q(z) = e_q(zq^{-N/2})\cdot$, $N = z\partial_z$.

The two-dimensional quantum algebra $E_q(2)$ is defined by the following commutation relations:

$$[J_3, J_\pm] = \pm J_\pm \tag{3}$$

$$[J_+, J_-] = 0 \tag{4}$$

The Casimir operator is given by $C = J_+ J_-$.

Now we define the q-differential representation of $E_q(2)$ as

$$J_3 = D_q \tag{5}$$

$$J_+ = E_q(y)\{\partial_x - 1/xD_q\} \tag{6}$$

$$J_- = e_q(-y)\{-\partial_x - 1/xD_q\} \tag{7}$$

where D_q is the q-derivation. It is defined by

$$D_q f(z) = \frac{f(zq^{1/2}) - f(zq^{-1/2})}{z(q^{1/2} - q^{-1/2})} = z^{-1}[N]_q f(z) \tag{8}$$

with

$$D_q e_q^{\alpha z} = \alpha e_q^{\alpha z} \tag{9}$$

$$D_q y^n = [n]_q y^{n-1} \tag{10}$$

It is straightforward to prove that the differential representations (5)–(7) generate the algebra (3) and (4). Hence the Casimir operator is given by

$$C_q = J_+J_- = -\partial_x^2 - 1/x\partial_x + 1/x^2 D_q^2 \tag{11}$$

We assume the eigenfunctions of C_q to be $e_q^{my}J_m(x, q)$, where $J_m(x,q)$ is the q-Bessel function. Then

$$C_q e_q^{my} J_m(x, q) = \lambda_q e_q^{my} J_m(x, q) \tag{12}$$

where λ_q is the eigenvalue of C_q.

From equations (11) and (12) one gets

$$J''_m(x, q) + 1/xJ'_m(x, q) + (\lambda_q - m^2/x^2)J_m(x, q) = 0 \tag{13}$$

We can choose λ_q such that the differential equation (13) has a polynomial solution.

We consider

$$J_m(x, q) = \sum_{n=0}^{\infty} c_n(q)x^{n \cdot r} \tag{14}$$

Then we get

$$c_n = \frac{\lambda_q}{m^2 - (n + r)^2} c_{n-2}, \qquad n \geq 2 \tag{15}$$

$$c_1 = 0, \qquad c_0 \neq 0 \tag{16}$$

If $r = r_1 = m$, then

$$J_m^{(1)}(x, q) = x^m \sum_{n=0}^{\infty} c_n(q)x^n \tag{17}$$

$$c_n(q) = \frac{\lambda_q}{m^2 - (n + m)^2} c_{n-2}, \qquad n \geq 2$$

and for $r = r_2 = - m$, one gets

$$J_m^{(2)}(x, q) = x^{-m} \sum_{n=0}^{\infty} c_n(q)x^n \tag{18}$$

$$c_n(q) = \frac{\lambda_q}{m^2 - (n - m)^2} c_{n-2}, \qquad n \geq 2 \tag{19}$$

The general form of the q-Bessel function is

$$J_m(x, q) = AJ_m^{(1)}(x, q) + BJ_m^{(2)}(x, q) \tag{20}$$

such that $r_1 - r_2 \notin Z \cup \{0\}$, where Z is the set of positive integers, and

$$e_q(xz/2)e_q(- x/2z) = \sum_{n=-\infty}^{\infty} z^n J_n^{(1)}(x, q) \tag{21}$$

$$E_q(xz/2)E_q(-qx/2z) = \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} z^n J_n^{(2)}(x, q) \tag{22}$$

The basis $e_q^{my} J_m(x, q)$ of the irreducible representation of $E_q(2)$ coincides with the irreducible representation of the quantum group $A(E_q(2))$, where $A(E_q(2))$ is the Hopf algebra generated by the elements z, $\bar{z}$, a, and $\bar{a}$ with the commutation relations

$$z\bar{z} = \bar{z}z = 1, \qquad a\bar{a} = \bar{a}a, \qquad za = qaz, \tag{23}$$
$$az = q\bar{z}a, \qquad \bar{a}\bar{z} = q\bar{z}\bar{a}, \qquad z\bar{a} = q\bar{a}z$$

where q is a real number.

The comultiplication is given by

$$\Delta(z) = z \otimes z \tag{24}$$

$$\Delta(a) = a \otimes 1 + z \otimes a \tag{25}$$

The counit and the antipode are given respectively by

$$\varepsilon(z) = 1, \qquad \varepsilon(a) = 0 \tag{26}$$

$$s(z) = \bar{z}, \qquad s(a) = -\bar{z}a \tag{27}$$

We can choose a quantum subalgebra $A(K)$ corresponding to translations in $A(E_q(2))$ which is defined as $C[t, \bar{t}]$. The coproduct Δ_K, counit ε_K, and antipode s_K are given by

$$\Delta_K(t) = t \otimes 1 + 1 \otimes t \tag{28}$$

$$\varepsilon_K(t) = 1, \qquad s_K(t) = -t \tag{29}$$

and the projection epimorphism

$$\pi: \qquad A(E_q(2)) \to C[t, \bar{t}] \tag{30}$$

is given by

$$\pi(a) = t, \qquad \pi(z) = 1 \tag{31}$$

The Hopf subalgebra $A(K)$ is coabelian and its irreducible representations are one-dimensional labeled by arbitrary complex numbers λ,

$$\rho_\lambda(w) = e^{\lambda t} \otimes w \tag{32}$$

where

$$\rho_\lambda: \qquad C \to A(K) \otimes C \tag{33}$$

Since the associative $A(E_q(2))$ generated by z, $\bar{z}$, a, and $\bar{a}$ defines the *-algebra structure which is defined by $z^* = \bar{z}$ and $a^* = \bar{a}$, we take the representation space H_λ for $\rho_\lambda \uparrow A(E_q(2))$ as the Hilbert space $A(E_q(2)) \otimes C \simeq A(E_q(2))$ of elements f satisfying

$$((id \otimes \pi) \circ \Delta) f = f \otimes e^{\lambda t} \tag{34}$$

where

$$(id \otimes \pi) \circ \Delta: \qquad A(E_q(2)) \to A(E_q(2)) \otimes A(K) \tag{35}$$

is the right coaction of $A(K)$ in $A(E_q(2))$.

This representations ρ_λ provide a family of irreducible representations for the quantum group $A(E_q(2))$ in the form $f \otimes e^{\lambda t}$ on a space of two complex variables z, a. Without loss of generality we take

$$f \otimes e^{\lambda t} \simeq F_\lambda(z, \pi\,(a)) \simeq J_\lambda(z) e^{\lambda a}$$

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